And one more limit related to e.

https://www.linkedin.com/feed/update/urn:li:activity:6639806920939184128

Calculate $\lim_{n\to\infty} n^{-n^2}\left((n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)\right)^n$.

Solution by Arkady Alt, San Jose ,California, USA.

Let $p_n := n^{-n^2}\prod_{k=1}^{n}\left(n+\frac{1}{2^{k-1}}\right)^n = \left(\prod_{k=1}^{n}\left(1+\frac{1}{n2^{k-1}}\right)\right)^n, n \in \mathbb{N}$ and $L := \lim_{n\to\infty} p_n$.

Noting that $x - \frac{x^2}{2} < \ln(1+x) < x, \forall x \in (0,1)$ and $\ln p_n = n\sum_{k=1}^{n}\ln\left(1+\frac{1}{n2^{k-1}}\right)$ we obtain

$$n \cdot \sum_{k=1}^{n}\left(\frac{1}{n2^{k-1}} - \frac{1}{n^2 2^{2k-1}}\right) < \ln p_n < n \cdot \sum_{k=1}^{n}\frac{1}{n2^{k-1}} \Leftrightarrow \sum_{k=1}^{n}\frac{1}{2^{k-1}} - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{2k-1}} < \ln p_n < \sum_{k=1}^{n}\frac{1}{2^{k-1}}$$

Since $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^{k-1}} = 2$ and $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{2k-1}} = 0$ then by Squeeze Principle $\lim_{n\to\infty}\ln p_n = 2$.

Hence, $L = \lim_{n\to\infty} p_n = e^{\lim_{n\to\infty}\ln p_n} = e^2$.